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ON CERTAIN SPECTRAL RELATIONSHIPS ASSOCIATED WITH THE CARLEMAN INTEGRAL EQUATION AND THEIR APPLICATIONS TO CONTACT PROBLEMS*

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Special relationships /1/ for an integral operator generated by a symmetric power series in a finite interval containing Gegenbauer polynomials is established again by methods of the theory of a generalized potential. Spectral relationships for this same operator are also established in the case of two symmetric semi-infinite intervals. A solution is constructed on the basis of these latter, for the contact problem of the impression of two identical semi-infinite stamps into a half-plane deformable according to a power law.

A number of similar spectral relationships in orthogonal polynomials for integral operators encountered in diverse mixed problems of elasticity theory and mathematical physics is established in papers for which a detailed bibliography is presented in /2/. A method, on their basis, using orthogonal polynomials which is developed substantially in these papers, permits an effective solution to be obtained for an extensive class of contact and mixed problems of the mechanics of a deformable body. The paper /3/ is also devoted to the application of the apparatus of orthogonal polynomials.

Carleman /4/ first considered an integral equation with a symmetric power series in a finite interval, where the method of continuation of the equation in the complex plane is used. A more general equation is examined in /5/ by the methods of the boundary value problems of the theory of analytic functions. A solution of the Carleman equation in the form of quadratures without integrals in the Cauchy sense was obtained first in /6/.

1. Let us consider the Carleman integral equation

$$\int_{-1}^{1} \frac{\varphi(s)\,ds}{|x-s|^h} = f(x), \quad 0<h<1, \quad \varphi(x) \in L_w^2(-1,1), \quad w(x)>0 \tag{1.1}$$

in order to determine the eigenfunctions and eigennumbers of the integral operator entering here. To this end, we consider the function of two variables

$$V(x,y) = \int_{-1}^{1} \frac{\varphi(s)\,ds}{[(x-s)^2+y^2]^{h/2}} \tag{1.2}$$

As is shown in /7,8/, the function $V(x,y)$ satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{h}{y}\frac{\partial V}{\partial y} = 0$$

everywhere in the $x0y$ plane except on the segment $L = \{-1 \leqslant x \leqslant 1,\ y=0\}$

For $h = {}^1/_3$ this equation is encountered also in problems of gasdynamics /9,10/ in connection with the known Tricomi problem. It is seen that

$$V(x,y) \sim \frac{P}{r^h}, \quad r\to\infty \quad (r=\sqrt{x^2+y^2}); \quad P = \int_{-1}^{1} \varphi(s)\,ds$$

Therefore, the solution of the integral equation (1.1) is equivalent to the solution of the following external boundary value problem

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$$\Delta V+\frac{h}{y}\frac{\partial V}{\partial y}=0,\quad (x,y)\overline{\in} L \tag{1.3}$$
$$V(x,y)|_{y=0}=f(x),\quad -1<x<1;\quad V(x,y)\sim\frac{P}{r^h},\quad r\to\infty$$

An analogous boundary value problem is obtained in /11/, where the question of uniqueness of the solution is touched upon.

After the solution of the problem (1.3) has been constructed, the source density will be determined by the formula

$$-2\sqrt{\pi}\,\Gamma[(1+h)/2]\,[\Gamma(h/2)]^{-1}\operatorname{sgn} y\varphi(x)=\lim_{y\to 0}|y|^h\frac{\partial V(x,y)}{\partial y} \tag{1.4}$$

We transform the boundary value problem (1.3) into an equivalent problem that allows application of the method of separation of variables, for which we set

$$V(x,\ y)=|y|^{-h/2}U(x,\ y)$$

Furthermore, as in /12/, by using the Zhukovskii function

$$z=\frac{1}{2}\left(\zeta+\frac{1}{\zeta}\right),\quad z=x+iy,\quad \zeta=\xi+i\eta=\rho e^{i\vartheta}$$

we map the plane z with a slit along the segment /-1,1/ on the unit circle $\rho<1$ of the plane ζ. We will have here

$$x=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\vartheta,\quad y=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\sin\vartheta \tag{1.5}$$

$$\frac{1}{4y^2}\left|\frac{dz}{d\zeta}\right|^2=\frac{1}{(\rho^2-1)^2}+\frac{1}{4\rho^2\sin^2\vartheta} \tag{1.6}$$

$$r=\frac{1}{2\rho}\sqrt{\rho^4+2\rho^2\cos 2\vartheta+1} \tag{1.7}$$

These results from (1.7) that the point $z=\infty$ corresponds to the point $\zeta=0$. Taking the latter and formulas (1.5) and (1.6) into account, we transform the boundary value problem (1.3) into the following boundary value problem for a unit circle:

$$\Delta W+h(2-h)\left[\frac{1}{(\rho^2-1)^2}+\frac{1}{4\rho^2\sin^2\vartheta}\right]W=0,\quad \rho<1 \tag{1.8}$$
$$\left[\frac{1}{2}\left|\left(\rho-\frac{1}{\rho}\right)\sin\vartheta\right|\right]^{-h/2}W(\rho,\vartheta)|_{\rho=1}=f(\cos\vartheta)$$
$$W(\rho,\vartheta)|_{\rho=0}=0,\quad -\pi\leqslant\vartheta<\pi$$
$$\left(\Delta W=\frac{\partial^2 W}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\vartheta^2}\right)$$
$$\left(W(\rho,\vartheta)=U\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\vartheta,\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\sin\vartheta\right]=U(x,y)\right)$$

We now apply the method of separation of variables to (1.8), for which we put

$$W(\rho,\ \vartheta)=R(\rho)\,\Phi(\vartheta)$$

We consequently arrive at the differential equations

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+\left[h(2-h)\frac{\rho^2}{(\rho^2-1)^2}-\lambda^2\right]R=0,\quad 0\leqslant\rho<1 \tag{1.9}$$

$$\frac{d^2\Phi}{d\vartheta^2}+\left[\lambda^2+\frac{h(2-h)}{4\sin^2\vartheta}\right]\Phi=0,\quad -\pi\leqslant\vartheta<\pi \tag{1.10}$$

where λ^2 is the separation parameter. By using (1.8) it is possible to detect that (1.9) should be examined under the condition

$$R(0)=0 \tag{1.11}$$

and the equation (1.10) under the periodicity condition

$$\Phi(\vartheta)=\Phi(\vartheta+2\pi) \tag{1.12}$$

Together with the additional condition cited below for (1.10), these conditions result in Sturm-Liouville boundary value problems for the differential equations mentioned.

Let us first construct the solution of (1.10) by considering $0 < \vartheta < \pi$. The substitution

$$\Phi(v) = \sqrt{\sin \vartheta} G(\vartheta), \ 0 < \vartheta < \pi$$

converts the equation mentioned into a Legendre differential equation that has the linearly independent solutions

$$P_\nu^\mu(\cos\vartheta), \ Q_\nu^\mu(\cos\vartheta); \ 0 < \vartheta < \pi, \ \nu = \lambda - 1/2, \ \mu = (1-h)/2$$

Representing the Legendre functions in the form of known trigonometric expansions /13/, by using (1.12) we find that there should be $\lambda = n + h/2$, $n = 0, 1, 2, \ldots$. From these expansions we find that the function $(\sin\vartheta)^\mu P_{n-\mu}^\mu(\cos\vartheta)$ $(0 < \vartheta < \pi)$ allows of an even continuation in the interval $-\pi < \vartheta < 0$ while the function $(\sin\vartheta)^\mu Q_{n-\mu}^\mu(\cos\vartheta)$ is an odd continuation of this same integral. But according to the second relationship (1.8), the desired solution should be even. Consequently, the single solution of equation (1.10) determined by means of (1.12) and this additional condition will have the form

$$\Phi(\vartheta) = \sqrt{|\sin\vartheta|}\, P_{n-\mu}^\mu(\cos\vartheta), \ -\pi \leqslant \vartheta < \pi, \ n = 0, 1, 2, \ldots \qquad (1.13)$$

Taking into account the relationship connecting Gegenbauer polynomials to Legendre functions /13/, we can finally write

$$\Phi(\vartheta) = |\sin\vartheta|^{h/2} C_n^{h/2}(\cos\vartheta), \ -\pi \leqslant \vartheta < \pi, \ n = 0, 1, 2, \ldots$$

Now turning to (1.9), we set

$$u = \rho^2, \ R(\sqrt{u}) = M(u), \ 0 \leqslant u < 1$$

Following the known procedure of the analytic theory of differential equations /14/, it can be shown that the equation obtained is determined by the following Riemann scheme

$$M(u) = u^{\lambda/2}(1-u)^{h/2} P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & h/2 & 0 & u \\ -\lambda & \lambda + h/2 & 1-h & \end{Bmatrix}$$

But the function $M_0(u)$ satisfying the hypergeometric differential equation /14/ for

$$a = h/2, \ b = \lambda + h/2, \ c = \lambda + 1, \lambda = n + h/2, \ n = 0, 1, 2, \ldots$$

is described by the Riemann scheme mentioned.

Linearly independent solutions of this equation will be

$$F(a, b; c; u), \ u^{1-c} F(a - c + 1, \ b - c + 1; \ 2 - c; \ u)$$

where $F(a, b; c; u)$ is the Gauss hypergeometric function. However, the second solution is not bounded at the point $u = 0$ for the parameters mentioned, and because of condition (1.11) only the first solution must be taken. Then finally

$$R(\rho) = \rho^{n+h/2}(1-\rho^2)^{h/2} F(h/2, \ n + h; \ n + 1 + h/2; \rho^2), \ 0 \leqslant \rho < 1 \qquad (1.14)$$

Comparison of (1.13) and (1.14) yields that the boundary value problem (1.3) possesses a normal solution of the form

$$V_0(\rho, \vartheta) = \rho^{n+h} F(h/2, n + h; n + 1 + h/2; \rho^2)\, C_n^{h/2}(\cos\vartheta) \qquad (1.15)$$

$$0 \leqslant \rho < 1, \quad -\pi \leqslant \vartheta < \pi, \quad n = 0, 1, 2, \ldots$$

$$\left(V_0(\rho, \vartheta) = V\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\vartheta, \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\vartheta\right] = V(x, y)\right)$$

Starting from the potential (1.15), we calculate the appropriate density of sources for which we transform (1.4). For definiteness, we obtain

$$\varphi(\cos\vartheta) = \frac{\Gamma(h)\,\Gamma(n + 1 + h/2)}{\sqrt{\pi}\, 2^{h/2}\,\Gamma[(1+h)/2]\,\Gamma(n+h)} (\sin\vartheta)^{h-1} C_n^{h/2}(\cos\vartheta), \ 0 < \vartheta < \pi \qquad (1.16)$$

by considering $0 < \vartheta < \pi$ and using (1.15).

Substituting (1.16) and (1.15) for $\rho = 1$ into (1.2), we obtain the desired spectral relationship after certain operations

$$\int_{-1}^{1} \frac{C_n^{h/2}(s)\, ds}{|x - s|^h (1 - s^2)^{(1-h)/2}} = \mu_n C_n^{h/2}(x), \quad |x| < 1, \quad n = 0, 1, 2 \ldots \qquad (1.17)$$

$$\mu_n = \pi\Gamma(n + h)\,[n!\,\Gamma(h)\cos(\pi h/2)]^{-1}$$

which had been established earlier by other methods /1/.

It is important to note that the method elucidated here permits obtaining also an expression for the integral from (1.17) outside the interval $|x|<1$, i.e., on the rays $|x|>1$. Namely, on these rays

$$\rho = |x| - \operatorname{sgn} x \sqrt{x^2-1}, \quad |x|>1$$

as results from (1.5) where the value $\vartheta=0$ corresponds to the ray $x>1$ and the value $\vartheta=\pi$ to the ray $x<-1$. Again, substituting (1.16) and (1.15) into (1.2), we arrive at an integral relation similar to (1.17) after manipulation ($H(x)$ is the Heaviside function)

$$\int_{-1}^{1} \frac{C_n^{h/2}(s)\,ds}{|x-s|^h (1-s^2)^{(1-h)/2}} = \nu_n [H(x) + (-1)^n H(-x)] [|x| - \operatorname{sgn} x \sqrt{x^2-1}]^{n+h} F(h/2, n+h; n+1+h/2; 2x^2 - 2x\sqrt{x^2-1} - 1), \quad |x|>1$$

$$\nu_n = \frac{\sqrt{\pi}\, 2^h \Gamma[(1+h)/2]\, \Gamma^2(n+h)}{\Gamma^2(h)\, n!\, \Gamma(n+1+h/2)}, \quad n=0,1,2,\ldots$$

2. We now turn to an examination of the integral equation

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \frac{\varphi(s)\,ds}{|x-s|^h} = f(x) \tag{2.1}$$

We again introduce a function of two variables

$$V(x,y) = \int_L \frac{\varphi(s)\,ds}{[(x-s)^2+y^2]^{h/2}} \quad (L = \{|x|>1, y=0\}) \tag{2.2}$$

where, as before, we assume the source density to possess the finite power

$$P = \int_L \varphi(s)\,ds < \infty$$

although this condition may not even be satisfied for individual harmonics. Then the integral equation (2.1) is equivalent to the following boundary value problem:

$$\begin{gathered} \Delta V + \frac{h}{y}\frac{\partial V}{\partial y} = 0, \quad (x,y) \overline{\in} L \\ V(x,y)|_{y=0} = f(x), \quad |x|>1, \quad V(x,y) \backsim \frac{P}{r^h}, \quad r\to\infty \end{gathered} \tag{2.3}$$

To construct the solution of (2.3), we note that the Zhukovskii function presented above maps the z plane with the cut off rays $|x|>1$ onto the $\eta>0$ half-plane of the ζ plane. The upper plane $y>0$ is here mapped onto an infinite semicircle $\{\rho>1,\ 0<\vartheta<\pi\}$ while the lower half-plane $y<0$ is mapped ont the semicircle $\{\rho<1,\ 0<\vartheta<\pi\}$. Therefore, (2.3) can be converted to the boundary value problem

$$\begin{gathered} \Delta W + h(2-h)\left[\frac{1}{(\rho^2-1)^2} + \frac{1}{4\rho^2 \sin^2\vartheta}\right] W = 0, \ \rho<\infty, \ 0<\vartheta<\pi \\ \left[\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\vartheta\right]^{-h/2} W(\rho,\vartheta)|_{\vartheta=0,\ \vartheta=\pi} = f\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\vartheta\right]_{\vartheta=0,\ \vartheta=\pi}, \quad 1<\rho<\infty \\ \left(\frac{1}{2}\left|\rho-\frac{1}{\rho}\right|\sin\vartheta\right)^{-h/2} W(\rho,\vartheta)|_{\rho\to 0,\ \rho\to\infty} \backsim \frac{P}{r^h}, \quad r\to\infty \end{gathered} \tag{2.4}$$

where the notation is as before.

Setting

$$W(\rho,\vartheta) = R(\rho)\,\Phi(\vartheta)$$

as before we again arrive from (2.4) at the differential equations (1.9) and (1.10) in which λ^2 should be replaced by $-\lambda^2$.

Furthermore, by proceeding perfectly analogously to the manner elucidated above, we find that the boundary value problem (2.3) possesses a normal solution of the form

$$V(x,y)=V_0(\rho,\vartheta)=\left(\frac{1}{2}\left|\rho-\frac{1}{\rho}\right|\sin\vartheta\right)^{\mu}\times \tag{2.5}$$
$$[AP_\nu^\mu(\cos\vartheta)+BQ_\nu^\mu(\cos\vartheta)]\left\{CP_\nu^\mu\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\right]+\right.$$
$$\left.DP_\nu^{-\mu}\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\right]\right\},\quad \rho<\infty,\quad 0<\vartheta<\pi$$
$$\nu=-{}^1/_2+i\lambda,\ \mu=(1-h)/2,\ \lambda>0$$

Now, starting from (2.5) we construct even and odd functions in ϑ with respect to the point $\vartheta=\pi/2$ We set

$$G(\vartheta)=AP_\nu^\mu(\cos\vartheta)+BQ_\nu^\mu(\cos\vartheta),\ 0<\vartheta<\pi$$

and we use the known relationships (/13/, p.145). Then the equality $G(\vartheta)=G(\pi-\vartheta)\ (0<\vartheta<\pi/2)$ governing the even function with respect to the point $\vartheta=\pi/2$, results in a linear homogeneous system with respect to the constants A and B. The determinant of this system is identically zero, and therefore, it has a nontrivial solution

$$B=\frac{2A}{\pi}\operatorname{tg}\left[\pi\left(\frac{h}{4}-\frac{i\lambda}{2}\right)\right]$$

By using this equality, we can represent the even real function that is a component of the normal solution by the formula

$$G_0(\vartheta)=(\sin\vartheta)^\mu G_\lambda^+(\vartheta),\quad 0<\vartheta<\pi \tag{2.6}$$
$$C_\lambda^+(\vartheta)=P_\nu^\mu(\cos\vartheta)+\frac{1}{\pi}\left\{\operatorname{tg}\left[\pi\left(\frac{h}{4}-\frac{i\lambda}{2}\right)\right]Q_\nu^\mu(\cos\vartheta)+\right.$$
$$\left.\operatorname{tg}\left[\pi\left(\frac{h}{4}-\frac{i\lambda}{2}\right)\right]Q_{\bar\nu}^\mu(\cos\vartheta)\right\}$$

We can analogously represent the odd real function by the formula

$$G_0(\vartheta)=(\sin\vartheta)^\mu G_\lambda^-(\vartheta),\quad 0<\vartheta<\pi \tag{2.7}$$
$$G_\lambda^-(\vartheta)=P_\nu^\mu(\cos\vartheta)-\frac{1}{\pi}\left\{\operatorname{ctg}\left[\pi\left(\frac{h}{4}-\frac{i\lambda}{2}\right)\right]Q_\nu^\mu(\cos\vartheta)+\right.$$
$$\left.\operatorname{ctg}\left[\pi\left(\frac{h}{4}+\frac{i\lambda}{2}\right)\right]Q_{\bar\nu}^\mu(\cos\vartheta)\right\}$$

Furthermore, we set $C=1$ and $D=0$ in (2.5). Then the normal solution of the boundary value problem (2.3) will have the form

$$V_0(\rho,\vartheta)=\left(\frac{1}{2}\left|\rho-\frac{1}{\rho}\right|\right)^\mu G_0(\vartheta)P_\nu^\mu\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\right] \tag{2.8}$$
$$\rho<\infty,\quad 0<\vartheta<\pi$$

If we set $C=0$ and $D=1$ in (2.5), we will then have

$$V_0(\rho,\vartheta)=\left(\frac{1}{2}\left|\rho-\frac{1}{\rho}\right|\right)^\mu G_0(\vartheta)P_\nu^{-\mu}\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\right] \tag{2.9}$$
$$\rho<\infty,\quad 0<\vartheta<\pi$$

Let us find the source density. Proceeding perfectly analogously to what was done above, we obtain from (2.8) and (2.6)

$$\varphi(x)=E_\lambda^h\left[\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\right]^\mu P_\nu^\mu\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\right], \tag{2.10}$$
$$x=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right),\quad \rho>1$$
$$E_\lambda^h=2^{-\mu}\pi^{-1/2}\Gamma(h/2)\frac{1+\cos(\pi h/2)\operatorname{ch}\pi\lambda}{\{|\Gamma(h/2+i\lambda)|[\operatorname{ch}\pi\lambda+\cos(\pi h/2)]\}^2}$$

We now go from the variable ρ to the variable x in (2.10) and (2.8) for $\vartheta=0$ and we substitute the results obtained into (2.2). After manipulation we arrive at the spectral relationship

$$\int_1^\infty\left[\frac{1}{|x-s|^h}+\frac{1}{(x+s)^h}\right](s^2-1)^{-\mu}\varphi_+(s,\lambda)\,ds=\sigma_+(\lambda)\varphi_+(x,\lambda),\quad x>1,\quad \lambda>0 \tag{2.11}$$

Starting from the other components of (2.6)—(2.7) and (2.8)—(2.9), by completely analogous means we obtain the following spectral relations as well:

$$\int_1^\infty \left[\frac{1}{|x-s|^h} - \frac{1}{(x+s)^h}\right](s^2-1)^{-\mu}\varphi_+(s,\lambda)\,ds = \sigma_-(\lambda)\varphi_+(x,\lambda), \quad x>1, \quad \lambda>0 \tag{2.12}$$

$$\int_1^\infty \left[\frac{1}{|x-s|^h} \pm \frac{1}{(x+s)^h}\right](s^2-1)^{-\mu}\varphi_-(s,\lambda)\,ds = \sigma_\pm(\lambda)\varphi_-(x,\lambda), \quad x>1, \quad \lambda>0 \tag{2.13}$$

$$(\varphi_\pm(x,\lambda) = (x^2-1)^{\mu/2}P_\nu^{\pm\mu}(x))$$

$$(\sigma_\pm(\lambda) = [\operatorname{ch}\pi\lambda \pm \cos(\pi h/2)]\,|\Gamma(h/2+i\lambda)|^2\,[\Gamma(h)\times \cos(\pi h/2)]^{-1})$$

By formal addition and subtraction of (2.11) and (2.12) as well as by (2.13), we arrive at still four other spectral relations

$$\int_1^\infty \frac{\varphi_+(s,\lambda)}{|x\pm s|^h}(s^2-1)^{-\mu}\,ds = \rho_\pm(\lambda)\varphi_+(x,\lambda), \quad x>1, \quad \lambda>0 \tag{2.14}$$

$$\int_1^\infty \frac{\varphi_-(s,\lambda)}{|x\pm s|^h}(s^2-1)^{-\mu}\,ds = \rho_\pm(\lambda)\varphi_-(x,\lambda), \quad x>1, \quad \lambda>0 \tag{2.15}$$

$$\rho_+(\lambda) = |\Gamma(h/2+i\lambda)|^2\,[\Gamma(h)]^{-1}, \quad \rho_-(\lambda) = \operatorname{ch}\pi\lambda\,|\Gamma(h/2+i\lambda)|^2\,[\Gamma(h)\cos(\pi\lambda/2)]^{-1}$$

Setting $h=1$ in (2.14) and (2.15) when there is a plus sign, we obtain the known Mellor spectral relationship /15/.

On the basis of the results elucidated, relationships similar to (2.11)—(2.13) can be obtained that are valid in the interval $0<x<1$. It can be noted that the arc $\{\rho=1, 0<\vartheta<\pi/2\}$ of a circle in the ζ plane corresponds to this interval. Proceeding analogously to the above, by using (2.8)—(2.9), (2.6) and (2.10), we arrive at the relations

$$\int_1^\infty \left[\frac{1}{|x-s|^h} \pm \frac{1}{(x+s)^h}\right](s^2-1)^{-\mu}\varphi_+(s,\lambda)\,ds = \pm\varkappa_\pm(\lambda)(1-x^2)^{\mu/2}G_\lambda^\pm(\arccos x), \quad 0<x<1 \tag{2.16}$$

$$\int_1^\infty \left[\frac{1}{|x-s|^h} \pm \frac{1}{(x+s)^h}\right](s^2-1)^{-\mu}\varphi_-(s,\lambda)\,ds = 0, \quad 0<x<1$$

$$\varkappa_\pm(\lambda) = |\Gamma(h/2+i\lambda)|^2 \frac{[\operatorname{ch}\pi\lambda \pm \cos(\pi h/2)]^2}{\{\Gamma(h)\,[1\pm\cos(\pi h/2)\operatorname{ch}\pi\lambda]\}}$$

Furthermore, by using the formulas for the generalized Mellor transform, we obtain the following bilinear expansion from (2.11):

$$\pi\Gamma(h)\cos\left(\pi\frac{h}{2}\right)\left[\frac{1}{|x-s|^h}+\frac{1}{(x+s)^h}\right][(x^2-1)(s^2-1)]^{-\mu/2} =$$
$$\int_0^\infty \lambda\operatorname{sh}\pi\lambda\left[\operatorname{ch}\pi\lambda+\cos\left(\frac{\pi h}{2}\right)\right]\left|\Gamma\left(\frac{h}{2}+i\lambda\right)\right|^4 P_\nu^\mu(x)P_\nu^\mu(s)\,d\lambda$$
$$x>1$$

It is seen that this last integral converges. The same expansions can be written by using (2.12)—(2.13).

3. We apply the results obtained to solve a contact problem on the impression of two identical semi-infinite stamps occupying the domain $\{|x|>a, y=0\}$ on a half-plane $y<0$ being deformed according to the power law $\sigma_i = K_0\varepsilon_i^\alpha$ $(0<\alpha<1)$.

We assume that the stamps subjected to the forces applied can be moved only translationally in the vertical direction. Here σ_i and ε_i are the stress and strain intensities, while K_0 and α are physical constants of the material. This physical law can be considered within both the framework of the deformation theory of platicity and the theory of steady creep, but in the latter case ε_i must be understood to be in the strain intensity. By adhering to the generalized principle of superposition of the displacements /16,17/, the solution of the problem mentioned can be reduced to the solution of the integral equation

$$\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) \frac{p(s)\,ds}{|x-s|^{1-\alpha}} = \left[\frac{\delta - f_0(x)}{A_0}\right]^{\alpha} \tag{3.1}$$

$$A_0 = (2-\gamma)\sin(\pi\beta/2)\,[K_0 J(\lambda)]^{-\gamma}\,[\beta(\gamma-1)]^{-1}$$

$$J(\alpha) = 4\int_0^{\pi/2} (\cos\beta\vartheta)^{\alpha}\cos\vartheta\,d\vartheta,\quad \beta = \sqrt{2\alpha-1}/\alpha,\quad \gamma = 1/\alpha$$

where $p(x)$ is the contact pressure, δ is the settlement of the stamps, and $f_0(x)$ is a function characterizing the surface of the stamps.

We limit ourselves to the examination of the symmetric case. As usual, we assume the contact pressure to have a finite resultant P. Then the left side of (3.1) has the asymptotic $P/|x|^{1-\alpha}$ as $|x|\to\infty$. Therefore

$$f_0(x) \sim \delta - A_0 P^{\gamma}|x|^{1-\gamma},\quad |x|\to\infty$$

from which the settlement δ of the stamp is actually determined.

Furthermore, we turn to dimensionless quantities in (3.1)

$$x = a\xi,\quad y = a\eta,\quad ap(a\xi)/P = \varphi(\xi),\quad a^{1-\alpha}A_0^{-\alpha}[\delta - f_0(a\xi)]^{\alpha}/P = f(\xi)$$

after which we will have the equation

$$\int_1^{\infty}\left[\frac{1}{|\xi-\eta|^h} + \frac{1}{(\xi+\eta)^h}\right]\varphi(\eta)\,d\eta = f(\xi),\quad h = 1-\alpha \tag{3.2}$$

We represent the solution of (3.2) in the form of the integral

$$\varphi(\xi) = (\xi^2-1)^{-\mu/2}\int_0^{\infty}\Phi(\lambda)\,P^{\mu}_{-1/2+i\lambda}(\xi)\,d\lambda \tag{3.3}$$

Taking (2.11) into account, we find by means of the Mellor inversion formula

$$\Phi(\lambda) = \frac{\Gamma(h)\cos(\pi h/2)\,\lambda\,\mathrm{sh}\,\pi\lambda}{\pi[\mathrm{ch}\,\pi\lambda + \cos(\pi h/2)]}\int_1^{\infty} P^{\mu}_{-1/2+i\lambda}(\xi)\,(\xi^2-1)^{-\mu/2} f(\xi)\,d\xi \tag{3.4}$$

Thus, the solution of (3.2) is given by (3.3) and (3.4).

Now, setting

$$v_0(\xi) = \int_1^{\infty}\left[\frac{1}{|\xi-\eta|^h} + \frac{1}{(\xi+\eta)^h}\right]\varphi(\eta)\,d\eta,\quad 0<\xi<1$$

by using the first relationship in (2.16) when the plus sign is taken, we obtain

$$v_0(\xi) = \varkappa_+(\lambda)\,(1-\xi^2)^{\mu/2}\int_0^{\infty} G^+_{\lambda}(\arccos\xi)\,\Phi(\lambda)\,d\lambda$$

Within the limits of the accuracy taken, the true displacements of the boundary points of the half-plane outside the stamps will be expressed by the formula

$$v(x) = -A_0 a^{1-\gamma} P^{\gamma} v_0^{\gamma}(x/a),\quad 0<x<a$$

Let us note that the results of this section can be extended to the problem under consideration in a linear elasticity theory formulation /18/ when the elastic modulus of the half-plane varies in depth according to the power law $E(y) = E_0|y|^{\alpha}$ $(0 \leqslant \alpha < 1,\ y<0)$.

It should still be noted that the application of potential theory methods will permit not only the establishment of a large number of spectral relationships known earlier, but also the obtaining of a number of new ones, and thereby, substantial broadening of their class. They afford the possibility of obtaining spectral relations by a single method for many integral operators.

Moreover, the necessary physical characteristics can be found not only in the domains where the integral equations are given, but also outside of them, which is especially important in three-dimensional problems.

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